(20) (a) $$I = \int \frac{x-7}{2x^2-3x-2}\,dx$$

Partial fractions: $$\frac{x-7}{2x^2-3x-2} = \frac{x-7}{(2x+1)(x-2)}$$

$$= \frac{A}{2x+1} + \frac{B}{x-2}$$

So $$x-7 = A(x-2) + B(2x+1)$$

when $x=2$: $\quad -5 = 5B \quad \Rightarrow B = -1$

$x = -\frac{1}{2}$: $\quad -\frac{15}{2} = -\frac{5}{2}A \quad \Rightarrow A = 3$

So

$$I = \int \frac{3}{2x+1} - \frac{1}{x-2}\,dx$$

$$= \frac{3}{2}\ln|2x+1| - \ln|x-2| + c$$

$$= \ln\frac{\sqrt{(2x+1)^3}}{x-2} + c$$

(b) $$I = \int_0^{\pi/3} \sin^3 x\,dx = \int_0^{\pi/3} \sin x \cdot \sin^2 x\,dx$$

$$= \int_0^{\pi/3} \sin x \cdot (1-\cos^2 x)\,dx$$

$$= \int_0^{\pi/3} \sin x - \sin x\cos^2 x\,dx$$

$$\therefore \quad I = \left[-\cos x + \frac{1}{3}\cos^3 x \right]_0^{\pi/3}$$

$$= \left(-\frac{1}{2} + \frac{1}{24} \right) - \left(-1 + \frac{1}{3} \right)$$

$$= \frac{5}{24}$$

(c) $I = \int_0^2 \sqrt{x(4-x)}\, dx$

let $x = 4\sin^2\theta$, $\therefore dx = 8\sin\theta\cos\theta\, d\theta$

And when $x = 0$, $\theta = 0$

$x = 2$, $\theta = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

So

$$I = \int_0^{\pi/4} \sqrt{4\sin^2\theta\,(4 - 4\sin^2\theta)} \cdot 8\sin\theta \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/4} \sqrt{4\sin^2\theta \cdot 4\cos^2\theta} \cdot 8\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/4} 4\sin\theta \cdot \cos\theta \cdot 8\sin\theta\cos\theta\, d\theta$$

$$= 32\int_0^{\pi/4} \sin^2\theta\cos^2\theta\, d\theta = 32\int_0^{\pi/4} \left(\frac{1}{2}\sin 2\theta\right)^2 d\theta$$

$$= 8\int_0^{\pi/4} \sin^2 2\theta\, d\theta$$

using $\cos 2\theta = 1 - 2\sin^2\theta$ we have
$\cos 4\theta = 1 - 2\sin^2 2\theta$

$$\therefore \quad I = 8\int_0^{\pi/4} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= 8\left[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta\right]_0^{\pi/4}$$

$$= \left(\pi - \sin\pi\right) - \left(0 - \frac{\sin 0}{8}\right)$$

$$= \pi$$

(21) (a) $$I = \int_0^{\pi/4} \sin 5x \cos 3x \, dx = \frac{1}{2}\int_0^{\pi/4} 2\sin 5x \cos 3x \, dx$$

$$2\sin 5x \cos 3x = 2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2} = \sin A + \sin B$$

So $\frac{A+B}{2} = 5x$ & $\frac{A-B}{2} = 3x$

$\Rightarrow$ $A = 8x$ & $B = 2x$

So $$I = \frac{1}{2}\int_0^{\pi/4} \sin 8x + \sin 2x \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(-\frac{1}{8}\cos 2\pi - \frac{1}{2}\cos\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{1}{8} - \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{8}\right) + \frac{1}{2}\left(\frac{1}{8}\right) + \frac{1}{4} = \frac{1}{4}.$$

(b) $I = \int_0^1 x \cdot e^{-3x}\, dx$

let $u = x$, $\therefore \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-3x}$, $\therefore v = -\frac{1}{3} e^{-3x}$

So $I = -\frac{x}{3} e^{-3x} \Big|_0^1 + \frac{1}{3}\int_0^1 e^{-3x}\, dx$

$= \left[-\frac{x}{3} e^{-3x} - \frac{1}{9} e^{-3x} \right]_0^1$

$= \left(-\frac{1}{3} e^{-3} - \frac{1}{9} e^{-3}\right) - \left(-\frac{1}{9}\right)$

$= \frac{1}{9}\left(1 - 4e^{-3}\right)$

(22) (a) $I = \int_0^{\pi/2} \sin 2x \cos 3x\, dx = \frac{1}{2}\int_0^{\pi/2} 2 \sin 2x \cos 3x\, dx$

$2 \sin 2x \cos 3x = 2 \cos \frac{A+B}{2} \cdot \sin \frac{A-B}{2} = \sin A - \sin B$

So $\frac{A+B}{2} = 3x$ & $\frac{A-B}{2} = 2x$

$\Rightarrow A = 5x$, $B = x$

$\therefore I = \frac{1}{2}\int_0^{\pi/2} \sin 5x - \sin x\, dx = \frac{1}{2}\left[\frac{-1}{5}\cos 5x + \cos x\right]_0^{\pi/2}$

$= -\frac{2}{5}$.

(b) $I = \int_0^{\pi/2} \sin^2 x \cdot \cos^2 x \, dx = \int_0^{\pi/2} (\sin x \cos x)^2 \, dx$

$$= \int_0^{\pi/2} \left(\frac{1}{2} \sin 2x\right)^2 dx = \int_0^{\pi/2} \frac{1}{4} \sin^2 2x \, dx$$

By $\cos 2x = 1 - 2\sin^2 x$ we have $\cos 4x = 1 - 2\sin^2 2x$

$$\therefore \quad I = \int_0^{\pi/2} \frac{1}{4} \left(\frac{1 - \cos 4x}{2} \right) dx$$

$$= \int_0^{\pi/2} \frac{1}{8} - \frac{1}{8} \cos 4x \, dx$$

$$= \left[\frac{1}{8} x - \frac{1}{32} \sin 4x \right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{16} - \frac{1}{32} \sin 2\pi \right) - 0 = \frac{\pi}{16}$$

(c) $I = \int_0^3 \frac{x}{\sqrt{25 - x^2}} \, dx$

let $u = 25 - x^2$, $\therefore \; du = -2x \, dx$

$$\Rightarrow -\frac{1}{2} \, du = x \, dx$$

And if $x = 0 \quad u = 25$

$x = 3 \quad u = 16$

So $I = \int_{25}^{16} -\frac{1}{2} \frac{1}{\sqrt{u}} \, du$

$$\therefore \quad I = \int_{16}^{25} \tfrac{1}{2} u^{-1/2}\, du = u^{1/2} \Big|_{16}^{25}$$

$$= 5 - 4 = 1$$

(d) $$I = \int_0^1 \frac{x}{6 - 5x + x^2}\, dx = \int_0^1 \frac{x}{(x-3)(x-2)}\, dx$$

Partial fractions: $$\frac{x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

So $$x = A(x-2) + B(x-3)$$

when $x = 2$: $2 = -B \Rightarrow B = -2$

$x = 3$: $3 = A$

So $$I = \int_0^1 \frac{3}{x-3} - \frac{2}{x-2}\, dx$$

$$\therefore \quad I = \Big[3 \ln|x-3| - 2 \ln|x-2| \Big]_0^1$$

$$= \left[\ln \left| \frac{(x-3)^3}{(x-2)^2} \right| \right]_0^1$$

$$= \ln \frac{2^3}{1} - \ln \frac{3^3}{2^2} = \ln 8 - \ln \frac{27}{4} = \ln \frac{32}{27}.$$

(23) (a) $I = \int_0^{\pi/4} \sin 3\theta \sin\theta \, d\theta = -\frac{1}{2}\int_0^{\pi/4} -2\sin 3\theta \sin\theta \, d\theta$

$$-2\sin 3\theta \cdot \sin\theta = -2\sin\frac{A+B}{2} \cdot \sin\frac{A-B}{2} \quad (*)$$

$$= \cos A - \cos B$$

From (*) : $\frac{A+B}{2} = 3\theta$; $\frac{A-B}{2} = \theta$

$\Rightarrow A = 4\theta$, $B = 2\theta$

So $I = \int_0^{\pi/4} -\frac{1}{2}\left(\cos 4\theta - \cos 2\theta\right) d\theta$

$$= -\frac{1}{2}\left(\frac{1}{4}\sin 4\theta - \frac{1}{2}\sin 2\theta\right)_0^{\pi/4}$$

$$= -\frac{1}{2}\left(\frac{1}{4}(0) - \frac{1}{2}(1)\right) + \frac{1}{2}(0-0)$$

$$= \frac{1}{4}.$$

(b) $I = \int \sin^2 x \cos^3 x \, dx = \int \sin^2 x(1-\sin^2 x)\cos x \, dx$

$$= \int \sin^2 x \cdot \cos x - \sin^4 x \cdot \cos x \, dx$$

let $u = \sin x$, $\therefore du = \cos x \, dx$

So $I = \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + c$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c.$$

(c) $I = \int \frac{x}{\sqrt{3+x}}\, dx$

let $u = 3+x$, $\therefore du = dx$

& $x = u-3$

So $I = \int \frac{u-3}{u^{1/2}}\, du = \int u^{1/2} - \frac{3}{u^{1/2}}\, du$

$$= \frac{2}{3}u^{3/2} - \frac{3}{2}u^{1/2} + c$$

$$= \frac{2}{3}(3+x)^{3/2} - \frac{3}{2}(3+x)^{1/2} + c.$$

(24) (a) $I = \int_1^{1.5} \frac{x+2}{(x+1)(x+3)}\, dx$

Partial fractions: $\frac{x+2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$\therefore \quad x+2 = A(x+3) + B(x+1)$

when $x=-1$: $1 = 2A \Rightarrow A = \frac{1}{2}$
$x=-3$: $-1 = -2B \Rightarrow B = \frac{1}{2}$

So $I = \int_1^{1.5} \frac{1/2}{x+1} + \frac{1/2}{x+3}\, dx$

$$= \left[\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x+3|\right]_1^{1.5}$$

$$I = \left[\frac{1}{2} \ln \left| (x+1)(x+3) \right| \right]_1^{1.5}$$

$$= \frac{1}{2} \ln \left(\frac{5}{2}\right)\left(\frac{9}{2}\right) - \frac{1}{2} \ln 8 = \frac{1}{2} \ln \frac{45}{32}$$

(b) $I = \int_0^{\pi/3} x . \sin 3x \, dx$

let $u = x$, $\therefore \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin 3x$, $\therefore v = -\frac{1}{3} \cos 3x$

$$\therefore I = \left[-\frac{x}{3} . \cos 3x \right]_0^{\pi/3} + \frac{1}{3} \int_0^{\pi/3} \cos 3x \, dx$$

$$= \left[-\frac{x}{3} \cdot \cos 3x \right]_0^{\pi/3} + \frac{1}{3} \left[\frac{1}{3} \sin 3x \right]_0^{\pi/3}$$

$$= \left[-\frac{\pi}{3}(-1) + \frac{1}{9}(0) \right] - \left[0 + 0 \right] = \frac{\pi}{3}$$

(c) $I = \int_{-1}^{2} x^2 . \sqrt{x^3+1} \, dx$

let $u = x^3 + 1$, $\therefore du = 3x^2 \, dx \Rightarrow \frac{1}{3} du = x^2 dx$

Also when $x = -1$, $u = 0$

$x = 2$, $u = 9$

$$\therefore \quad I = \int_0^9 \sqrt{u} \cdot du = \left[\frac{2}{3} u^{3/2} \right]_0^9 = 6.$$

(25) $x \cdot \frac{dy}{dx} = (1-2x^2) \cdot y$, when $x > 0$ & $y(1) = 1$

$$\therefore \quad \int \frac{1}{y}\, dy = \int \frac{1-2x^2}{x}\, dx$$

$$\therefore \quad \ln|y| = \int \frac{1}{x} - 2x \, dx = \ln|x| - x^2 + c$$

But $y = 1$ when $x = 1$, hence

$$\ln 1 = \ln 1 - 1 + c \Rightarrow c = 1$$

$$\therefore \quad \ln|y| = \ln|x| - x^2 + 1$$

$$\therefore \quad e^{\ln|y|} = e^{\ln|x| - x^2 + 1}$$

hence $|y| = |x| e^{1-x^2}$

Since x is positive we have $y = x \cdot e^{1-x^2}$, since y will also be positive

(26) (a)

$$I = \int_0^1 \frac{1-4x}{3+x-2x^2}\, dx = \int_0^1 \frac{1-4x}{(3-2x)(1+x)}\, dx$$

Partial fractions: $\frac{1-4x}{(3-2x)(1+x)} = \frac{A}{3-2x} + \frac{B}{1+x}$

$\therefore \quad 1-4x = A(1+x) + B(3-2x)$

when $x=-1$: $5 = 5B \Rightarrow B=1$

$x = \frac{3}{2}$: $-5 = 2\frac{1}{2}A \Rightarrow A=-2$

So $I = \int_0^1 \frac{-2}{3-2x} + \frac{1}{1+x}\, dx$

$= \Big[\ln|3-2x| + \ln|1+x|\Big]_0^1$

$= (\ln 1 + \ln 2) - (\ln 3 + \ln 1) = \ln \frac{2}{3}$

(b) $I = \int x . e^{2x}\, dx$

let $u = x$, $\therefore \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{2x}$, $\therefore v = \frac{1}{2}e^{2x}$

So $I = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\, dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$

$= \frac{1}{4}(2x-1)e^{2x} + c$

(c) $\frac{dy}{dx} = xy$ for $y(0) = 2$.

So $\int \frac{1}{y}\, dy = \int x\, dx \Rightarrow \ln|y| = \frac{x^2}{2} + c$

By $y(0) = 2$ we have

$$\ln 2 = 0 + c \Rightarrow c = \ln 2$$

So $\ln|y| = \frac{x^2}{2} + \ln 2$

Either $|y| = e^{x^2/2 + \ln 2} = 2e^{x^2/2}$

or $\ln|y| - \ln 2 = \frac{x^2}{2} \Rightarrow x^2 = 2\ln\left|\frac{y}{2}\right|$

(27) $(1+\cos 2x)\frac{dy}{dx} - (1+e^y)\sin 2x = 0$ for $y(\frac{\pi}{4}) = 0$.

$$\therefore \quad (1+\cos 2x)\frac{dy}{dx} = (1+e^y)\sin 2x$$

So $\int \frac{1}{1+e^y}\,dy = \int \frac{\sin 2x}{1+\cos 2x}\,dx$

Multiply LHS by $\frac{e^{-y}}{e^{-y}}$ to get

$$\int \frac{e^{-y}}{e^{-y}+1}\,dy = \int \frac{\sin 2x}{1+\cos 2x}\,dx$$

$$\therefore \quad -\ln|e^{-y}+1| = -\frac{1}{2}\ln|1+\cos 2x| + c$$

But $y = 0$ when $x = \frac{\pi}{4}$ so

$$-\ln 2 = -\frac{1}{2}\ln 1 + c \Rightarrow c = -\ln 2$$

Hence

$$-\ln|e^{-y}+1| = -\frac{1}{2}\ln|1+\cos 2x| - \ln 2$$

$$\ln|e^{-y}+1| = \ln 2\sqrt{1+\cos 2x}$$

But $\cos 2x = 2\cos^2 x - 1$, $\therefore$ $1+\cos 2x = 2\cos^2 x$

Hence $\ln|e^{-y}+1| = \ln 2\sqrt{2\cos^2 x} = \ln 2\sqrt{2}\cdot\cos x$

$$\therefore \quad e^{-y}+1 = 2\sqrt{2}\cdot\cos x$$

(28) $(3x+5)^2\frac{dy}{dx} = \frac{1+4y^2}{1+y}$ for $y(0)=0$.

So $\displaystyle\int \frac{1+y}{1+4y^2}\,dy = \int \frac{1}{(3x+5)^2}\,dx$

$\therefore$ $\displaystyle\int \frac{1}{1+4y^2} + \frac{y}{1+4y^2}\,dy = \int \frac{1}{(3x+5)^2}\,dx$ (*)

For $\displaystyle\int \frac{1}{1+4y^2}\,dy$ we substitute $y = \frac{1}{2}\tan\theta$. Therefore $dy = \frac{1}{2}\sec^2\theta\,d\theta$

Hence $\displaystyle\int \frac{1}{1+4y^2}\,dy = \int \frac{1}{1+\tan^2\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta$

$$= \int \frac{1}{2}\,d\theta = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}(2y) + c_1$$

For $\int \frac{y}{1+4y^2}\,dy$ let $u = 1+4y^2$.

$\therefore\ du = 8y\,dy \Rightarrow \frac{1}{8}\,du = y\,dy$

Hence $\int \frac{y}{1+4y^2}\,dy = \int \frac{1}{8}\cdot\frac{1}{u}\,du = \frac{1}{8}\ln|u| + c_2$

$= \frac{1}{8}\ln|1+4y^2| + c_2$

For $\int \frac{1}{(3x+5)^2}$ let $u = 3x+5$, $\therefore\ du = 3\,dx$

$\Rightarrow \frac{1}{3}\,du = dx$

So $\int \frac{1}{(3x+5)^2}\,dx = \int \frac{1}{3}\frac{1}{u^2}\,du = -\frac{1}{3}\cdot\frac{1}{u} = -\frac{1}{3}\cdot\frac{1}{3x+5} + c_3$

Therefore, by $(*)$ we have

$$\frac{1}{2}\tan^{-1}(2y) + \frac{1}{8}\ln|1+4y^2| = -\frac{1}{3}\cdot\frac{1}{3x+5} + k$$

where $k = c_3 - c_1 - c_2$.

Now, $y=0$ when $x=0$, $\therefore\ \frac{1}{2}(0) + \frac{1}{8}(0) = -\frac{1}{3}\cdot\frac{1}{5} + k$

$\Rightarrow k = \frac{1}{15}$

So we have $\frac{1}{2}\tan^{-1}(2y) + \frac{1}{8}\ln|1+4y^2| = -\frac{1}{3}\cdot\frac{1}{3x+5} + \frac{1}{15}$

(29) (a) i) let $I = \int_0^1 \frac{1+x}{1+2x}\,dx = \int_0^1 \frac{1}{1+2x} + \frac{x}{1+2x}\,dx$

Now, $\int_0^1 \frac{1}{1+2x}\,dx = \frac{1}{2}\ln(1+2x)\Big|_0^1 = \frac{1}{2}\ln 3 - 0 = \frac{1}{2}\ln 3$

For $\int_0^1 \frac{x}{1+2x}\,dx$, let $u = 1+2x$, $\therefore \frac{u-1}{2} = x$

and $\frac{1}{2}\,du = dx$

Also if $x=0$, $u=1$ & if $x=1$, $u=3$

$\therefore \int_0^1 \frac{x}{1+2x}\,dx = \int_1^3 \frac{1}{2}\cdot\left(\frac{u-1}{2}\right)\cdot\frac{1}{u}\,du$

$= \int_1^3 \frac{1}{4}\cdot\frac{u-1}{u}\,du = \frac{1}{4}\int_1^3 1 - \frac{1}{u}\,du$

$= \frac{1}{4}\Big(u - \ln u\Big)\Big|_1^3 = \left(\frac{3}{4} - \frac{\ln 3}{4}\right) - \left(\frac{1}{4} - \frac{\ln 1}{4}\right)$

$= +\frac{1}{2} - \frac{\ln 3}{4}$

$\therefore I = \frac{1}{2}\ln 3 + \frac{1}{2} - \frac{\ln 3}{4} = +\frac{1}{2} + \frac{1}{4}\ln 3$

ii) let $I = \int_0^{\pi/2} \sin x \cdot \cos^2 x\,dx$.

let $u = \cos x$, $\therefore du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$

Also if $x=0$, $u=1$; if $x=\pi/2$, $u=0$.

$$\therefore \quad I = -\int_1^0 u^2\,du = \int_0^1 u^2\,du = \frac{1}{3}u^3\Big|_0^1$$

$$= \frac{1}{3}$$

(b) For $y = 2x - x^2$ we have

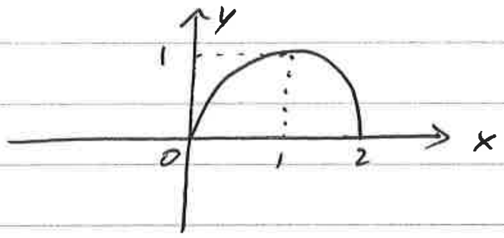
$$2x - x^2 = x(2-x)$$

So function crosses x axis at $x=0$ & $x = 2$.

Max occurs at $x = \frac{-b}{2a}$ $\Rightarrow$ max is at $x = -\frac{2}{-2} = 1$

& $y = 1$

$\therefore$ Relevant are is



Hence $A = \int_0^2 2x - x^2\,dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2$

$$= \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}.$$

(30) Let $(1+x^2)\frac{dy}{dx} - y(y+1)x = 0$; $y(0) = 1$

$$\therefore \quad (1+x^2)\frac{dy}{dx} = y(y+1)x \Rightarrow \int \frac{1}{y(y+1)}\,dy = \int \frac{x}{1+x^2}\,dx$$

For the RHS let $u = 1+x^2$, $\therefore$ $du = 2x\,dx$

$\Rightarrow \frac{1}{2}\,du = x\,dx$

$$\therefore \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C_1$$

$$= \frac{1}{2}\ln|1+x^2| + C_1$$

For the LHS use partial fractions:

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{1+y}$$

$$\therefore \quad 1 = A(1+y) + By$$

if $y=0$: $1 = A$

if $y=-1$: $1 = -B \Rightarrow B = -1$

$$\therefore \int \frac{1}{y(y+1)}\,dy = \int \frac{1}{y} - \frac{1}{y+1}\,dy$$

$$= \ln|y| - \ln|y+1| + C_2$$

So we have $\frac{1}{2}\ln|1+x^2| = \ln|y| - \ln|y+1| + k$, $k = C_2 - C_1$

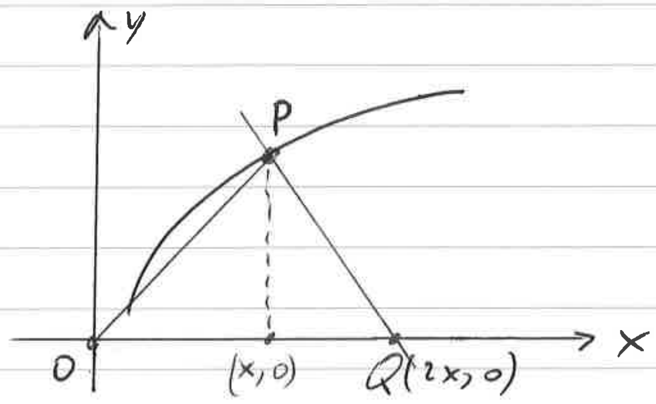
$$= \ln\left|\frac{y}{y+1}\right| + k$$

Since $y=1$ when $x=0$, $\frac{1}{2}\ln 1 = \ln\left(\frac{1}{2}\right) + k \Rightarrow k = -\ln\frac{1}{2}$.

So $\frac{1}{2}\ln|1+x^2| = \ln\left|\frac{y}{1+y}\right| - \ln\frac{1}{2} = \ln\left|\frac{2y}{1+y}\right|$

So $\sqrt{1+x^2} = \frac{2y}{1+y}$.

(31) Based on The Question we have the following Sketch:



$P(x,y)$ & $Q(2x,0)$

are The coordinates of P & Q.

So gradient of Normal PQ is $\frac{y_1 - y_0}{x_1 - x_0} = \frac{0 - y}{2x - x} = -\frac{y}{x}$

$\therefore$ gradient of tangent at P is $\frac{dy}{dx} = \frac{x}{y}$, $\therefore \; y \cdot \frac{dy}{dx} = x$

$\therefore \quad \int y \, dy = \int x \, dx$

$\Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + c$, i.e $y^2 = x^2 + A$ where $A = 2c$

(32) let $\frac{dy}{dx} = \frac{x(y^2-1)}{y(x^2+1)}$; $y(1) = 3$.

Then $\int \frac{y}{y^2-1}\,dy = \int \frac{x}{x^2+1}\,dx$ (*)

For LHS use partial fractions:

$$\frac{y}{y^2-1} = \frac{y}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}$$

So $y = A(y-1) + B(y+1)$

let $y=1$: $1 = 2B \Rightarrow B = \frac{1}{2}$

$y=-1$: $-1 = -2A \Rightarrow A = \frac{1}{2}$

$$\therefore \int \frac{y}{y^2-1}\,dy = \int \frac{\frac{1}{2}}{y+1} + \frac{\frac{1}{2}}{y-1}\,dy$$

$$= \frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| + C_1$$

$$= \frac{1}{2}\ln|(y+1)(y-1)| + C_1$$

For RHS of (*) use $u = x^2+1$, $\therefore du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

Hence $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C_2$

$$= \frac{1}{2}\ln|x^2+1| + C_2$$

$\therefore \frac{1}{2}\ln|y^2-1| = \frac{1}{2}\ln|x^2+1| + k$; $k = C_2 - C_1$

Since $y=3$ when $x=1$ we have

$$\frac{1}{2}\ln 8 = \frac{1}{2}\ln 2 + k \quad \Rightarrow \quad k = \frac{1}{2}\ln 4$$

So
$$\frac{1}{2}\ln|y^2-1| = \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\ln 4$$

$$y^2-1 = 4(x^2+1)$$

$$\therefore \quad y^2 = 4x^2+5.$$

(33) (a) given $\quad (1+x)\frac{dy}{dx} = (1-x)y, \quad y(0)=1$

we have
$$\int \frac{1}{y}\,dy = \int \frac{1-x}{1+x}\,dx.$$

$$= \int \frac{1}{1+x} - \frac{x}{1+x}\,dx$$

$$= \int \frac{1}{1+x} - \left(1 - \frac{1}{1+x}\right)dx,$$

by long division.

$$\therefore \quad \ln|y| = \ln|1+x| - x + \ln|1+x| + C$$

$$= \ln|(1+x)^2| - x + c$$

but $x=0,\ y=1,\ \therefore\ 0 = 0 - 0 + c \Rightarrow c=0$

$$\therefore \quad \ln|y| = \ln|(1+x)^2| - x = \ln|(1+x)^2| + \ln e^{-x}$$

$$\therefore \quad y = (1+x)^2 \cdot e^{-x}$$

(b) This is about finding y from the differential equation knowing $x=2$, $y=2$.

Hence from $\quad y e^{y^2} \frac{dy}{dx} = e^{2x}$

we have

$$\int y e^{y^2}\, dy = \int e^{2x}\, dx$$

$$\therefore \quad \tfrac{1}{2} e^{y^2} = \tfrac{1}{2} e^{2x} + c$$

Since $x=2$, $y=2$, we have $\quad \tfrac{1}{2} e^{4} = \tfrac{1}{2} e^{4} + c$

$$\Rightarrow \quad c = 0$$

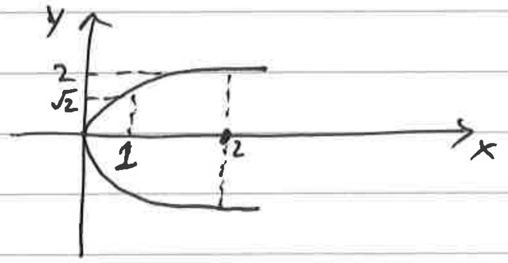
$$\therefore \quad e^{y^2} = e^{2x} \Rightarrow y^2 = 2x$$

For the point $(1, \sqrt{2})$: $\quad (\sqrt{2})^2 = 2(1)$

$$\therefore \quad 2 = 2 \quad \checkmark$$

So the curve passes through $(1, \sqrt{2})$.

Sketch



(34) If $\frac{dy}{dx} = \frac{y(y+1)}{x(x+1)}$, $y(1) = 2$

then $\int \frac{1}{y(y+1)} dy = \int \frac{1}{x(x+1)} dx$

use partial fractions: $\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$

$\therefore \quad 1 = A(y+1) + By$

If $y=0$: $1 = A$
$y=-1$: $B = -1$

The same coefficients apply to the partial fraction split of the RHS.
So

$$\int \frac{1}{y} - \frac{1}{y+1} dy = \int \frac{1}{x} - \frac{1}{x+1} dx$$

So $\ln|y| - \ln|y+1| = \ln|x| - \ln|x+1| + c$

But $x=1$, $y=2$, $\therefore$ $\ln 2 - \ln 3 = \ln 1 - \ln 2 + c$
$\therefore \ln \frac{4}{3} = c$

So $\ln|y| - \ln(y+1) = \ln|x| - \ln|x+1| + \ln \frac{4}{3}$

$$\ln\left|\frac{y}{y+1}\right| = \frac{4}{3} \ln\left|\frac{x}{x+1}\right|$$

(35) (a) i) let $I = \int \frac{x^2}{x+2}\,dx$

By long division:

$$\begin{array}{r|l} & x \quad -2 \\ \hline x+2 & x^2 \\ & x^2 + 2x \\ \hline & \quad -2x \\ & \quad -2x - 4 \\ \hline & \qquad +4 \end{array}$$

$\therefore \quad I = \int x - 2 + \frac{4}{x+2}\,dx$

$= \frac{x^2}{2} - 2x + 4\ln|x+2| + C$

ii) let $I = \int \sin 3x \cdot \cos 2x\,dx = \frac{1}{2}\int 2\sin 3x \cdot \cos 2x\,dx$

Now by trig identity: $2\sin 3x \cdot \cos 2x = 2\sin\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$

So $\frac{A+B}{2} = 3x$ & $\frac{A-B}{2} = 2x$

$\Rightarrow A = 5x \quad , \quad B = x$

So $I = \frac{1}{2}\int \sin 5x + \sin x\,dx = \frac{1}{2}\cdot\frac{1}{5}\cos 5x + \frac{1}{2}\cdot\cos x + C$

$= \frac{1}{10}(\cos 5x + 5\cos x) + c.$

(b) For intersection we have

$$e^x = 2 + 3e^{-x}$$

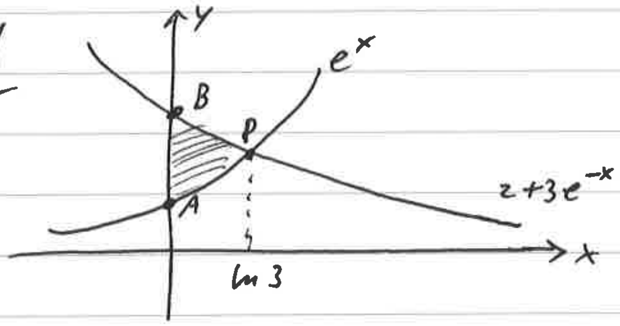
$$\therefore \quad e^{2x} = 2e^x + 3 \quad \Rightarrow e^{2x} - 2e^x - 3 = 0$$

Solve as a quadratic in e^x : $e^x = \dfrac{2 \pm \sqrt{4+12}}{2}$

$$e^x = +3 \quad \& \quad e^x = -1$$

But $e^x = -1$ is not valid, $\therefore e^x = 3 \Rightarrow x = \ln 3, \ y = 3.$

Sketch



$A = e^0 = 1,$
$B = 2 + 3e^0 = 5.$

So Area $\quad A = \displaystyle\int_0^{\ln 3} 2 + 3e^{-x} - e^x \, dx$

$$= \left[2x - 3e^{-x} - e^x \right]_0^{\ln 3}$$

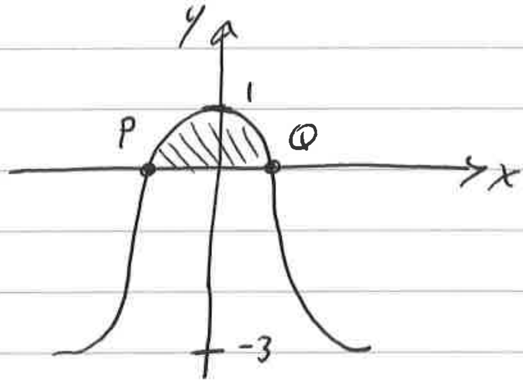
$$= \left(2\ln 3 - 3e^{-\ln 3} - e^{\ln 3}\right) - (-3 - 1)$$

$$= 2\ln 3 - 1 - 3 + 3 + 1 \quad = 2\ln 3$$

(36) For $y = 2\cos x - 1$ Solve $0 = 2\cos x - 1$;

$\therefore \quad \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, -\frac{\pi}{3}$ are the only two values in $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Sketch



where $P = -\frac{\pi}{3}$ & $Q = \pi/3$

So Area $A = \int_{-\pi/3}^{\pi/3} 2\cos x - 1 \, dx$

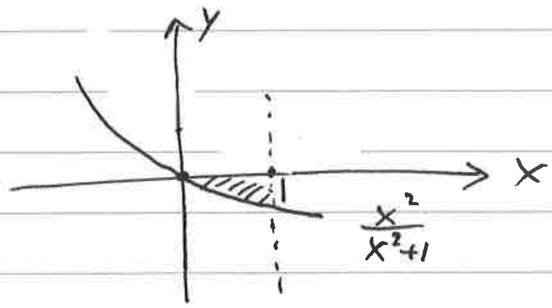
$= \Big[2\sin x - x\Big]_{-\pi/3}^{\pi/3} = \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(-2\sin\frac{\pi}{3} + \frac{\pi}{3}\right)$

$= 4\sin\frac{\pi}{3} - \frac{2\pi}{3}.$

$= 2\sqrt{3} - 2\frac{\pi}{3}$.

(37) By long division: $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$

Sketch:



So Area $A = \int_0^1 1 - \frac{1}{x^2+1} \, dx$

$$A = \Big[x - \tan^{-1} x \Big]_0^1$$

$$= (1 - \tan^{-1} 1) - (0 - \tan^{-1} 0)$$

$$= - \tan^{-1} 1 = -\frac{\pi}{4}$$

So the area is $\pi/4$ (& negative sign means that the area lies underneath the x-axis)

(38) $y = x(x-a)(x-b)$

$$= x^3 - x^2(a+b) + abx$$

$$\therefore y' = 3x^2 - 2x(a+b) + ab = m \quad (m = \text{gradient})$$

For a straight line : $y - y_0 = m(x - x_0)$

$$\therefore y = m(x - x_0) + y_0$$

So $\quad 4x = ab(x-0) + 0 \quad$ at $(0,0)$ ①

& $\quad 3 - 3x = (3a^2 - 2a(a+b) + ab)(x-a) + 0 \quad$ at $(a,0)$ ②

By ① : $ab = 4$ (*)

By ② : $(a^2 - ab)(x-a) = 3 - 3x \Rightarrow (a^2 - 4)(x-a) = 3 - 3x$ by ①.

$$\therefore a^2 x - a^3 - 4x + 4a = 3 - 3x$$

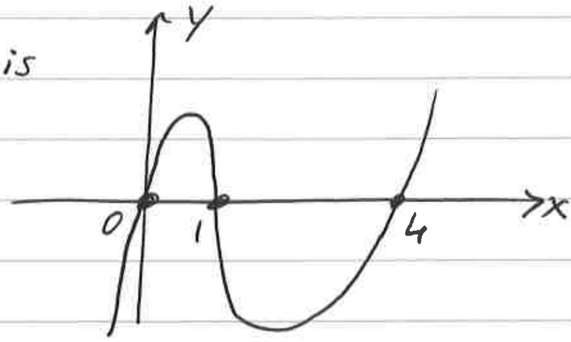
Comparing coeffs : xs : $a^2 - 4 = -3 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

Const : $-a^3 + 4a = 3$. If $a = 1$: $-1 + 4 = 3$ ✓

$a = -1$: $1 - 4 = -3$ ✗

So $a = 1 \Rightarrow b = 4$ by (*)

Hence $y = x(x-1)(x-4)$. Sketch is



So area $A = \int_1^4 x^3 - 5x^2 + 4x \, dx$

$= \left[\frac{x^4}{4} - \frac{5}{3}x^3 + 2x^2 \right]_1^4$

$= \left(\frac{4^4}{4} - \frac{5}{3}4^3 + 32 \right) - \left(\frac{1}{4} - \frac{5}{3} + 2 \right)$

$= -11\frac{1}{4}$

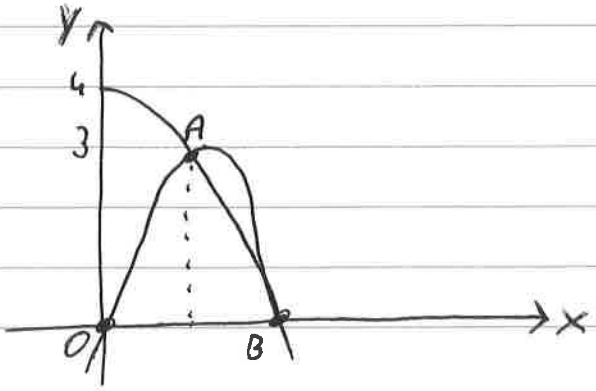
So Area is 11.25 units^2.

(39) For $y = 3\sin x$ & $y = 4\cos x$ we have

$3\sin x = 4\cos x \Rightarrow \tan x = \frac{4}{3}$

$\therefore x = \tan^{-1}\frac{4}{3} = 53.13° = 0.927$ radians.

Sketch



Area $A = \int_0^{0.927} 3\sin x \, dx + \int_{0.927}^{\pi/2} 4\cos x \, dx$

$$A = \Big[-3\cos x\Big]_0^{0.927} + \Big[4\sin x\Big]_{0.927}^{\pi/2}$$

$$= \big(-3(0.6)+3\big) + \big(4(1) - 4(0.8)\big)$$

$$= 1.2 + 0.8 = +2$$

So area A is 2 square units.

(40) (a) given $\frac{1}{y}\frac{dy}{dx} - x = xy$, $y(0)=1$

we have $\frac{1}{y}\frac{dy}{dx} = x + xy = x(1+y)$

$\therefore$

$$\int \frac{1}{y(1+y)}\,dy = \int x\,dx$$

Partial fractions: $\frac{1}{y(1+y)} = \frac{A}{y} + \frac{B}{1+y}$

$\therefore \quad 1 = A(1+y) + By$

If $y=0$: $\quad 1 = A$

$y=-1$: $\quad 1 = -B \Rightarrow B=-1$

$\therefore$ we have $\int \frac{1}{y} - \frac{1}{1+y}\,dy = \int x\,dx$

$$\Rightarrow \quad \ln|y| - \ln|1+y| = \frac{x^2}{2} + c$$

But $y=1$ when $x=0$, so

$$\ln 1 - \ln 2 = 0 + c \quad \Rightarrow \quad c = -\ln 2$$

$$\therefore \quad \ln|y| - \ln|1+y| = \frac{x^2}{2} - \ln 2$$

$$= \ln e^{x^2/2} - \ln 2$$

$$= \ln 2e^{x^2/2}$$

So $$\frac{y}{1+y} = 2e^{x^2/2}$$

$$\therefore \quad y = 2e^{x^2/2}(1+y) = 2e^{x^2/2} + y.2e^{x^2/2}$$

$$\Rightarrow \quad y(1-2e^{x^2/2}) = 2e^{x^2/2}$$

$$\therefore \quad y = \frac{2e^{x^2/2}}{1-2e^{x^2/2}} = \frac{2e^{x^2/2}}{1-2e^{x^2/2}} \cdot \frac{\frac{1}{2}e^{-x^2/2}}{\frac{1}{2}e^{-x^2/2}}$$

$$= \frac{1}{\left(\frac{1}{2}e^{-x^2/2} - 1\right)}$$

(b) If $x - y = z$ Then $1 - \frac{dy}{dx} = \frac{dz}{dx}$

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{dz}{dx}$$

FOR $\frac{dy}{dx} = x - y$, $y(0) = 0$, let $x - y = z$ (*)

$$\therefore \quad 1 - \frac{dz}{dx} = z \quad \Rightarrow \quad \frac{dz}{dx} = 1 - z$$

$$\therefore \quad \int \frac{1}{1-z}\,dz = \int dx$$

Also, when $y=0$ if $x=0$, we have $z=0$ when $x=0$ by (*)

$$\therefore \quad -\ln|1-z| = x + c$$

By $z=0$ when $x=0$ we have $-\ln 1 = 0 + c \Rightarrow c = 0$

$$\therefore \quad \ln \frac{1}{1-z} = x$$

$$\therefore \quad \frac{1}{1-z} = e^{x} \Rightarrow 1-z = e^{-x}$$

$$\Rightarrow y - x + 1 = e^{-x}$$

$$\therefore \quad y = x + e^{-x} - 1$$

(41) "The sum of masses is constant" implies $x + y = k_1$ (*)

"The time rate of change of x is proportional to the product of the masses" implies

$$\frac{dx}{dt} \propto x \cdot y, \text{ hence } \frac{dx}{dt} = k_2\, x \cdot y$$

$$= k_2\, x(k_1 - x) \text{ by (*)}$$

$$= k\, x\,(a - x)$$ by the notation of the question.

The constant "a" is just the sum of the two masses, given as k_1 in (*)

So $\displaystyle\int \frac{1}{x(a-x)}\,dx = \int k\,dt$

Partial fractions: $\displaystyle\frac{1}{x(a-x)} = \frac{A}{x} + \frac{B}{a-x}$

$\therefore \quad 1 = A(a-x) + Bx$

$x=0: \ 1 = aA \Rightarrow A = \frac{1}{a}$

$x=a: \ 1 = Ba \Rightarrow B = \frac{1}{a}$

$\therefore \displaystyle\int \frac{1}{a}\cdot\frac{1}{x} + \frac{1}{a}\frac{1}{a-x}\,dx = \int k\,dt$

$\Rightarrow \quad \frac{1}{a}\ln|x| - \frac{1}{a}\ln|a-x| = kt + c$

But $x = \frac{a}{10}$ when $t=0$, $\therefore$

$\frac{1}{a}\ln\frac{a}{10} - \frac{1}{a}\ln\frac{9a}{10} = c \quad = \frac{1}{a}\ln\frac{1}{9}$

So $\quad \frac{1}{a}\ln\left|\frac{x}{a-x}\right| = kt + \frac{1}{a}\ln\frac{1}{9} \qquad (**)$

By $x+y=a \Rightarrow y = a-x \Rightarrow \frac{a}{10} = a-x$, $\therefore x = \frac{9a}{10}$.

Into $(**)$: $\displaystyle\frac{1}{a}\ln\left|\frac{9a/10}{a-\frac{9a}{10}}\right| = kt + \frac{1}{a}\ln\frac{1}{9}$

$\therefore \displaystyle\frac{1}{ka}\left(\ln\left|\frac{9a/10}{a/10}\right| - \ln\frac{1}{9}\right) = t \Rightarrow \frac{1}{ka}\ln 9^2 = t \Rightarrow \frac{2}{ka}\ln 9 = t.$